Nuclear Sizes and Shapes

The **Charge distribution** within a nucleus can be determined using **elastic electron scattering**.

If we assume a uniform distribution of protons within the volume of the nucleus, the charge distribution can give an approximation to the distribution of matter within the nucleus.

Elastic electrons scattering, is, to a first approximation, just like Rutherford Scattering. The incident charge is \(-e\) instead of \(+2e\) and mass is different.

However, we must now use **quantum mechanics** and **relativity** in the derivation of the scattering formula. We will first write down the result assuming the electron is scattering from a **point charge**.
If the nucleus can be considered a point charge, the electron scattering cross section is

$$\frac{d\sigma}{d\Omega_{Mott}} = \frac{Z^2\alpha^2\hbar^2c^2}{4p^2v^2\sin^4(\theta/2)} \left[ 1 - \frac{v^2}{c^2}\sin^2(\theta/2) \right]$$

This is known as the Mott cross section.

**Note:**

$p = \text{momentum}$ of the electron. (Not distance of closest approach.)

$v = \text{speed}$ of the electron.

$\alpha = \text{fine structure constant} \approx 1/137$

This has been derived using quantum mechanics and relativity.

It includes the effects of the electron spin.

It assumes a spin-less nucleus.

The Mott cross section is exactly like the Rutherford scattering formula except for the additional part in the square brackets.
Electron Scattering

But the nucleus does have a size:

The actual scattering of an electron will be the sum of the scatterings from each piece of the nucleus.
i.e. from each volume element, $dV$, in the nucleus.

We need to treat the electron as a quantum mechanical particle.
Electron Scattering

We can write the Mott cross section as

$$\frac{d\sigma}{d\Omega}_{\text{Mott}} = |Z_{\text{ef}}(\theta)|^2$$

Charge of the scattering centre

A function of $\theta$

Scattering amplitude $A(\theta)$

This is like writing:

Probability $\propto |\text{Electron wave function}|^2$

For a finite nucleus:

$$\frac{d\sigma}{d\Omega} = \left| \sum_i A_i(\theta) \right|^2$$

Amplitude for scattering from each volume element $dV$. 
Electron Scattering

Define: Charge density within the nucleus \( = \rho(\vec{r}) \)

We will consider scattering from a volume element at position \( S \) compared to scattering from centre \( C \).

\( S \) can be specified by the spherical polar coordinates \( (r, \alpha, \beta) \)

(We use \( \alpha, \beta \) so as not to confuse with \( \theta \).)
Electron Scattering

Charge at S:
\[ dQ = \rho(\mathbf{r}) \, dV \]
\[ = \rho(\mathbf{r}) r^2 \sin \alpha \, dr \, d\alpha \, d\beta \]

The scattering amplitude from S is then:
\[ = dQ f(\theta) e^{i\delta} \]

where: \( e^{i\delta} \) is the phase amplitude difference compared to scattering from the centre C.

\[ \delta = \text{phase angle} = \frac{2\pi d}{\lambda} \quad (\text{can be related to } \theta \text{ and } \mathbf{r}) \]

\[ d = \text{path length difference} \]
\[ \lambda = \text{De Broglie wavelength of electron.} \]

Electron momentum:
\[ p = |\mathbf{p}_i| = |\mathbf{p}_f| = \frac{h}{\lambda} \]
Electron Scattering

We define the **Momentum Transfer**:

\[ \vec{q} = - (\vec{p}_f - \vec{p}_i) \]

\( \vec{q} \) = momentum transferred from the electron to the nucleus.

\[ q = |\vec{q}| = 2p \sin(\theta / 2) \]

Some geometry (next page) can show that

\[ \delta = \frac{2\pi d}{\lambda} = \frac{\vec{q} \cdot \vec{r}}{\hbar} \]

So scattering amplitude from S is \( \rho(\vec{r})dVf(\theta)e^{i\vec{q} \cdot \vec{r} / \hbar} \).
Electron Scattering

Derivation of

\[ \delta = \frac{2\pi d}{\lambda} = \frac{\mathbf{q} \cdot \mathbf{r}}{\hbar} \]

\[ p = |\mathbf{p}_i| = |\mathbf{p}_f| = \frac{h}{\lambda} \]

Additional path length

\[ d = \frac{\mathbf{p}_i \cdot \mathbf{r}}{p} - \frac{\mathbf{p}_f \cdot \mathbf{r}}{p} \]

Therefore:

\[ \delta = \frac{2\pi d}{\lambda} = \frac{2\pi (\mathbf{p}_i - \mathbf{p}_f) \cdot \mathbf{r}}{\lambda p} = \frac{2\pi \mathbf{q} \cdot \mathbf{r}}{\lambda (h/\lambda)} = \frac{\mathbf{q} \cdot \mathbf{r}}{\hbar} \]
Electron Scattering

Therefore the scattering amplitude from the whole nucleus is

\[ A(\theta) = \int_{V} \rho(\vec{r}) dV f(\theta) e^{i \vec{q} \cdot \vec{r} / \hbar} = f(\theta) \int_{V} \rho(\vec{r}) e^{i \vec{q} \cdot \vec{r} / \hbar} dV \]

\[ = f(\theta) \int_{\beta=0}^{\infty} \int_{\alpha=0}^{\pi} \int_{r=0}^{\infty} \rho(\vec{r}) e^{i \vec{q} \cdot \vec{r} / \hbar} r^2 \sin \alpha d\alpha d\beta dr \]

The total charge in the nucleus is

\[ Ze = \int_{V} \rho(\vec{r}) dV \]

\[ = \int_{\beta=0}^{\infty} \int_{\alpha=0}^{\pi} \int_{r=0}^{\infty} \rho(\vec{r}) r^2 \sin \alpha d\alpha d\beta dr \]
Electron Scattering

So we can write:

\[ A(\theta) = Z_{ef} (\theta) \frac{\int \rho(\vec{r}) e^{i\vec{q} \cdot \vec{r} / \hbar} dV}{Z_e} \]

Amplitude for scattering from a point charge \( Z_e \)

The total differential cross section is:

\[ \frac{d\sigma}{d\Omega} = |A(\theta)|^2 \]

\[ = |Z_{ef} (\theta)|^2 |F(\theta)|^2 \]

\[ = |F(\theta)|^2 \frac{d\sigma}{d\Omega}_{\text{Mott}} \]

Modification by the nuclear charge distribution

\[ F(\theta) = \frac{1}{Z_e} \int_{V} \rho(\vec{r}) e^{i\vec{q} \cdot \vec{r} / \hbar} dV \]
Electron Scattering

\( F(\theta) \) is more properly a function of \( q \).

\( q \) depends on both \( p \) and \( \theta \).

So it is more commonly written

\[
F(q^2) = \frac{1}{Ze} \int_V \rho(\vec{r}) e^{i\vec{q} \cdot \vec{r} / \hbar} dV
\]

\( F(q^2) \) is called a Form Factor.

It represents the modification of the point particle scattering by the finite nuclear size.
Electron Scattering

If the Nucleus is Spherically Symmetric.

i.e. \( \rho(\mathbf{r}) = \rho(r) \)

then

\[
F(q^2) = \frac{4\pi \hbar}{Z_{eq}} \int \rho(r) r \sin(qr / \hbar) dr
\]

Note: as \( q \to 0 \) (\( \theta \to 0 \)) then \( F(q^2) \to 1 \)

i.e. we have point charge like behavior.

as \( q \) increases, \( F(q^2) \) decreases
**Electron Scattering**

From a point charge \( Ze \):

\[
\frac{d\sigma}{d\Omega_{\text{Mott}}} = \frac{Z^2 \alpha^2 \hbar^2 c^2}{4 p^2 v^2} \frac{1}{\sin^4(\theta/2)} \left[ 1 - \frac{v^2}{c^2} \sin^2(\theta/2) \right]
\]

From a charge distribution \( \rho(r) \):

\[
\frac{d\sigma}{d\Omega} = \left| F(q^2) \right|^2 \frac{d\sigma}{d\Omega_{\text{Mott}}}
\]

\[
F(q^2) = \frac{1}{Ze} \int_V \rho(\vec{r}) e^{i\vec{q} \cdot \vec{r} / \hbar} dV
\]

\( F(q^2) \) is the **Form Factor**.

It is essentially the Fourier transform of the charge distribution.

If the Nucleus is **Spherically Symmetric**.

i.e. \( \rho(\vec{r}) = \rho(r) \)

then

\[
F(q^2) = \frac{4\pi \hbar}{Ze_{\text{eq}}} \int \rho(r) r \sin(qr / \hbar) dr
\]
To investigate the effects of nuclear size we will take a simple “Toy Model”.

We will assume the charge distribution is spherically symmetric and has a hard edge. i.e. \( \rho(r) = \text{constant, } r < a \) and \( \rho(r) = 0, r > a \)

We can show that: 
\[
F(q^2) = \frac{3}{x^3} \left( \sin x - x \cos x \right) \quad \text{with} \quad x = \frac{qa}{\hbar}
\]

Note: if \( x \) is small, 
\[
F(q^2) \approx 1 - \frac{x^2}{10} \quad \text{so} \quad F(q^2) \to 1 \quad \text{as} \quad q \to 0
\]

This also lets us see why we write \( F(q^2) \) rather than \( F(\theta, p) \).
Electron Scattering

Toy Model

\[ F(q^2) = \frac{3}{x^3} (\sin x - x \cos x) \]

with \( x = \frac{qa}{\hbar} \)

Actually:

\[ F(q^2) = \frac{3}{x} J_1(x) \]

\( J_1(x) \) is a Bessel function:

So:

\[ \left| F(q^2) \right|^2 \]

log scale

First zero is at \( x = 4.5 \)
Electron Scattering

Toy Model

If we take a nucleus with \( a \approx 4.5 \text{ fm} \)
The first zero is when,
\[
\frac{qa}{\hbar} = 4.5
\]
\[
\Rightarrow \frac{q}{\hbar} = \frac{4.5}{4.5} \text{ fm}^{-1} \approx 1 \text{ fm}^{-1}
\]

Often momenta are expressed in “inverse Fermi” \( \text{fm}^{-1} \) units (really \( \frac{q}{\hbar} \))

So \( q = \hbar(1 \text{ fm}^{-1}) = \hbar c(1 \text{ fm}^{-1}) / c \)

Then using \( \hbar c = 197 \text{ MeV.fm} \)

First minimum is at \( q = 197 \text{ MeV}/c \)
Electron Scattering

Toy Model

So: \( |F(q^2)|^2 \)

\[ a \approx 4.5 \text{ fm} \]

Use \( q = 2p \sin(\theta/2) \) to relate to \( \theta \).

If incident electron energy = 450 MeV
Electron Scattering

In practice an experiment measures the differential cross section.

\[ \frac{d\sigma}{d\Omega} = |F(q^2)|^2 \frac{d\sigma}{d\Omega}_{\text{Mott}} \]

Measure

In principle it is possible to invert

\[ F(q^2) = \frac{1}{Ze} \int_V \rho(\vec{r}) e^{i\vec{q} \cdot \vec{r} / \hbar} dV \]

to find \( \rho(\vec{r}) \)

e.g. for \(^{208}\text{Pb}\) with spherical symmetry assumed.
Electron Scattering

Usually a model for $\rho(r)$ is assumed and then this is used to obtain a best fit to $F(q^2)$ data.

A practical model is

$$\rho(r) = \frac{\rho_0}{1 + e^{\left(\frac{r-a}{d}\right)}}$$

This is called a **Woods-Saxon** shape.

The best fit to many nuclei gives,

$$a = 1.18A^{1/3} - 0.48 \text{ fm}$$

with $d \approx 0.55 \text{ fm}$, for $A > 40$.

This is the **charge radius**.

Usually the nuclear radius given by $R = 1.2A^{1/3}$ is pretty close.
Electron Scattering

More detailed fits can reveal some internal structure in the charge distribution.

e.g.
Optical Model

Electron scattering reveals the charge distribution in nuclei. The nuclear matter distribution can be probed by scattering neutrons.

In this case the nuclear force is involved. Charged particles (e.g. protons or pions) can be used as well. This, however, introduces the extra complication of both the nuclear force and the electromagnetic interaction acting together.

The main difference between using neutrons instead of electrons as a probe, is that we do not only get elastic scattering. We also get absorption of the neutron by the nuclear matter.

A model known as the optical model is used to interpret neutron scattering experiments.
The Optical Model:

- Describes the nucleus as a potential well with the ability to absorb.
- This is in analogy with optics for light scattering through a not completely transparent glass.
- The model is sometime called the “cloudy crystal ball” model.
- The model has many parameters!
- The results for a Wood-Saxon shaped potential well gives $a = 1.2A^{1/3}$ fm, $d = 0.75$ fm.

Same as for electron scattering.
Larger than for the charge distribution ($d \approx 0.55$ fm)